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New Formulation of Restricted Growth Processes

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We present a new formulation of a class of growth models–those which evolve according to an exclusion process. This formulation is based upon a transformation of the probability distribution function which involves Grassmann variables. This method is very general and enables one to derive an exact stochastic differential equation for the model of interest. We describe this method using the "traffic" model as an example.

KEY WORDS: Exclusion process; interface growth.

1. INTRODUCTION

There has been enormous interest in recent years concerning models of interface growth.[1] Such models include the Eden model,[2] ballistic deposition,[3] restricted solid-on-solid (RSOS) models,[4] and diffusion-limited aggregation (DLA).[5] Most theoretical investigations have been centered around a phenomenological Langevin equation for the evolution of the interface height h which is often called the KPZ (Kardar, Parisi, and Zhang) equation[6]:

$$\partial_t h(\mathbf{x}, t) = \nabla^2 h(\mathbf{x}, t) + \lambda(\nabla h(\mathbf{x}, t))^2 + \xi(\mathbf{x}, t) \tag{1.1}$$

where ξ is a Gaussian-distributed white noise. In $(1+1)$ dimensions the predictions from an RG analysis of this equation are in excellent agreement with simulations; i.e., there is a scaling regime in which the interface width obeys the scaling relation

$$w(t) \sim L^{\chi} f(t/L^{z}) \tag{1.2}$$

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where L is the linear system size. For intermediate times such that $t \ll L^z$ we have $w(t) \sim t^\beta$ where $\beta = \chi/z$. The exponents have the values[6] $z = 3/2$ and $\chi = 1/2$, which indicate that the interface width grows as $\omega(t) \sim t^{1/3}$ for $t \ll L^z$.

The case of more practical interest is $(2+1)$ dimensions. Here the situation is less clear. The RG analysis[6] of the KPZ equation indicates that the coupling constant λ is marginal in this case, and no perturbative fixed point exists. (It is worth mentioning that if the system enters a strong-coupling regime, then the relevance of terms not present in the original KPZ equation is brought into question—this is one of the primary motivations for this present work.) A mapping to a directed polymer in a random medium[7] indicates that the exponents retain their values found in the $(1+1)$ case, thus leading to claims of "superuniversality." However, simulations of various ballistic deposition models[4,8] have indicated that the interface width grows in $(2+1)$ dimensions with $\beta \sim 1/4$. It is worth noting that in simulations in $(2+1)$ dimensions, the exponents are exceedingly difficult to obtain. In order to find β, one must be in a regime where $1 \ll t \ll L^z$, indicating that one must use both very large systems and also run simulations for very long times. The situation is further complicated in $(2+1)$ dimensions with predictions of an asymptotically smooth interface[8] (based on a mapping of the single-step RSOS model to a six-vertex spin model), the announcement of $\beta \sim 0.13$ (from simulations of an asymmetric SOS model and also from integration of the KPZ equation[9]), and also the discovery of parameter-dependent exponents.[10] There is always a possibility that the asymptotic behavior neither follows a power law nor has universal characteristics.

If we are to have a better understanding of universality classes in these problems, we need a more systematic way of analyzing the large number of models of interface growth. In this article we shall set forth such a method whereby an exact Langevin equation may be derived for a given growth model. We should stress that we shall not be making predictions of exponents in this article. Our purpose is to show possible directions of future research by presenting a general framework in which to describe different growth models.

The method to be presented may be applied to any stochastic growth process which satisfies the following two conditions. First the underlying stochastic process must be Markovian, such that a master equation may be constructed. Second, if we consider the growth to proceed on a lattice, then the occupation of a given lattice site must be either zero or unity, i.e., we have an exclusion process. (It is then convenient to use the set of occupation numbers to describe a given state of the system. In a case such as the RSOS model it is possible to use the neighboring height differences along

the interface as the fundamental variables if they are taken to have only two possible values.) We mention that some previous attempts to construct a stochastic theory for growth processes[11] have relaxed the second condition by allowing multiple occupation and introducing a fast elimination process for multiply-occupied sites. We wish the theory to resemble as closely as possible the growth rules (usually based upon an exclusion process) used in simulations, and the effect of the above-mentioned alteration is unclear.

The outline of this paper is as follows. In the next section we briefly describe the traffic model which we shall use throughout this paper to demonstrate our general method. In Section 3 we give a thorough discussion of a new formulation for restricted growth processes (exclusion processes). This is developed for the traffic model for ease of presentation. In Section 4 we present a solution to a special case of this model using the newly developed formalism. We end the paper with Section 5, which is devoted to a summary of the work, along with our conclusions.

2. TRAFFIC MODEL

We have stressed the generality of our approach, but for purposes of presentation we shall consider a specific example, namely the "traffic" model. This model is in fact equivalent to the single-step RSOS model in $(1+1)$ dimensions.[8] We consider a one-dimensional chain whose sites may be either vacant or occupied. Define $\{x_i\}$ as the set of occupation numbers for the chain, where the subscript i denotes the position along the chain and $x_i = 0, 1$. A particle at a given site has a probability to hop either left or right (with rate constants k_L and k_R) so long as the neighboring site in question is vacant. If the neighboring site is occupied, then no transition is allowed—this is the exclusion process. Defining $P(\{x_i\}; t)$ as the probability distribution for a given set of occupation numbers $\{x_i\}$ at time t, we have the following master equation:

$$\begin{aligned}\partial_t P(\{x_i\}; t) = \frac{k}{2} \sum_i \{&(1-a)[x_{i-1}(1-x_i)\, P(\{..., x_{i-1}-1, x_i+1, ...\}; t) \\ &- x_i(1-x_{i-1})\, P(\{x_i\}; t)] \\ &+ (1+a)[x_{i+1}(1-x_i)\, P(\{..., x_i+1, x_{i+1}-1, ...\}; t) \\ &- x_i(1-x_{i+1})\, P(\{x_i\}; t)]\}\end{aligned} \tag{2.1}$$

In the above equation we have set $k_L = (k/2)(1-a)$ and $k_R = (k/2)(1+a)$, where $-1 \leqslant a \leqslant 1$. So k is a fundamental rate constant and a controls the

amount of bias in the system. From this equation it is possible to derive the (intractable) hierarchy of equations of motion for correlation functions by standard methods.

3. A NEW FORMULATION OF EXCLUSION PROCESSES

It would be preferable to have a Fokker–Planck-type description corresponding to (2.1). This is clearly not possible in the language of $P(\{x_n\}; t)$ due to the discrete and restricted range of the occupation numbers. A technique exists which allows an exact Fokker–Planck equation to be derived from a master equation, so long as the stochastic variables can take all integer values. This technique is based upon the so-called Poisson transformation introduced by Gardiner and Chaturvedi.[12] Although the stochastic variables in (2.1) are restricted in their allowed values, one may use a transformation based upon the Poisson transformation to derive an exact Fokker–Plank equation.

To achieve this, we introduce at each site two Grassmann variables[13]: (α_i, β_i). As usual, any two variables selected from $\{\alpha_i, \beta_i\}$ anticommute; and the square of any given variable is zero. We define differentiation and integration in the usual manner:

$$\partial_\alpha \cdot 1 = 0, \quad \partial_\alpha \cdot \alpha = 1; \quad \int d\alpha \cdot 1 = 0, \quad \int d\alpha \cdot \alpha = 1 \tag{3.1}$$

Since we shall always be dealing with the bilinear forms $\alpha_i \beta_i$ we introduce for convenience the variables $\{\gamma_i\}$: $\gamma_i \equiv \alpha_i \beta_i$. Evidently these new variables mutually commute although their square is zero. Defining $\partial_\gamma \equiv \partial_\alpha \partial_\beta$ and $\int d\gamma \equiv \int d\alpha \int d\beta$, we have the following rules for differentiation and integration on $\{\gamma_n\}$:

$$\partial_\gamma \cdot 1 = 0, \quad \partial_\gamma \cdot \gamma = -1; \quad \int d\gamma \cdot 1 = 0, \quad \int d\gamma \cdot \gamma = -1 \tag{3.2}$$

We now define a quasi-probability distribution $f(\{\gamma_i\}; t)$ via the transformation

$$P(\{x_i\}; t) = \int \prod_r d\gamma_r \, e^{-\gamma_r} \gamma_r^{x_r} f(\{\gamma_i\}; t) \tag{3.3}$$

For convenience we shall use $f(\gamma; t)$ as an alternative notation for $f(\{\gamma_i\}; t)$. We see that

$$\sum_{\{x_i\}} P(\{x_i\}; t) = 1 = \int \prod_r d\gamma_r f(\gamma; t) \tag{3.4}$$

This provides the "normalization" condition for the quasi-probability function $f(\gamma; t)$.

Before applying this transformation to (2.1) we shall discuss the relations between averages over P and "averages" over f, defined via

$$\langle G(\gamma)\rangle_f \equiv \int \prod_r d\gamma_r\, G(\gamma) f(\gamma; t) \tag{3.5}$$

One finds the following "connection formulas":

$$\langle x_i\rangle_P = \langle \gamma_i\rangle_f \tag{3.6}$$

$$\langle x_i x_j\rangle_P = \langle \gamma_i\gamma_j\rangle_f; \qquad i \neq j \tag{3.7}$$

and so on; if the average considered contains no monomials (of order higher than one) of x_i, then there is a one-to-one correspondence between the averages over the two distributions. Averages involving higher-order monomials are easily handled, since $x_i = 0, 1$ implying $x_i^p = x_i$. This means that

$$\langle \ldots x_i^p \ldots\rangle_P = \langle \ldots x_i \ldots\rangle_P = \langle \ldots \gamma_i \ldots\rangle_f, \qquad p \geqslant 1 \tag{3.8}$$

This is consistent since

$$\langle \ldots \gamma_i^p \ldots\rangle_f = 0, \qquad p > 1 \tag{3.9}$$

due to the Grassmann character of the $\{\gamma_i\}$.

A positive feature of (3.3) is that the transform is invertible and the function f is unique. To make this clear, we may represent the quasi-distribution function f as a sum of all possible polynomials of different γ's,

$$f(\gamma; t) = \sum a_{i_1,\ldots,i_n} \prod \gamma_{i_1} \cdots \gamma_{i_n} \tag{3.10}$$

When calculating a given moment or, in other words, a γ-product on average, the integration in (3.3) gives a nonzero result only for one of the terms in (3.10), namely for the term which forms the complete set of all γ's on all sites when multiplied by the γ-product. The corresponding coefficient $a_{i_1,\ldots,i_n}$ is therefore the moment of the P-distribution, $\langle x_{j_1},\ldots, x_{j_m},\ldots\rangle$, so that the two sets of subscripts $i_1,\ldots, i_n$ and $j_1,\ldots, j_n,\ldots$ form the complete set.

We would like to insert (3.3) into (2.1) and manipulate the terms such that an equation of motion may be written for $f(\gamma; t)$. This procedure is possible given the following relations [valid for arbitrary $f(\gamma; t)$]:

$$\int d\gamma_n\, e^{-\gamma_n\gamma_n^{x_n}} x_n f(\gamma; t) = \int d\gamma_n\, e^{-\gamma_n\gamma_n^{x_n}} (1 - \partial_{\gamma_n})\, \gamma_n f(\gamma; t) \tag{3.11}$$

and

$$\int d\gamma_n\, e^{-\gamma_n}\gamma_n^{x_n-1}x_n f(\gamma;t)$$

$$= -\int d\gamma_n\, e^{-\gamma_n}\gamma_n^{x_n}(1-\partial_{\gamma_n})\,\gamma_n(1+\partial_{\gamma_n})\,f(\gamma;t) \tag{3.12}$$

Given these relations, we may derive the following Fokker–Planck-type equation from (2.1):

$$\partial_t f(\gamma;t) = -\sum_i \partial_{\gamma_i} A_i(\gamma) f(\gamma;t) + \tfrac{1}{2}\sum_{i,j}\partial_{\gamma_i}\partial_{\gamma_j} B_{i,j}(\gamma) f(\gamma;t) \tag{3.13}$$

where the drift vector $A_i(\gamma)$ has the form

$$A_i(\gamma) = -\gamma_i + \tfrac{1}{2}(1+a)\,\gamma_{i-1} + \tfrac{1}{2}(1-a)\,\gamma_{i+1} + a\gamma_i(\gamma_{i+1}-\gamma_{i-1}) \tag{3.14}$$

and the diffusion matrix is given by

$$B_{i,j}(\gamma) = \gamma_i\gamma_j(\delta_{i,j-1}+\delta_{i,j+1}) + B_{i,i}\delta_{i,j} \tag{3.15}$$

It is important to realize at this stage that the diagonal part of the diffusion matrix is arbitrary, since in the Fokker–Planck equation it is operated on by $\partial_{\gamma_i}\partial_{\gamma_i}$ which is identically zero.

One can check at this stage that equations of motion for correlation functions calculated from (i) the master equation and (ii) the Fokker–Planck equation for f (with subsequent use of the connection formulas) are in agreement. It is worth noting that method (ii) for calculating these equations is far more efficient.

We would like to go one stage further in this stochastic formulation of exclusion processes. Given the Fokker–Planck equation for $f(\gamma;t)$, it is natural to derive from this a stochastic differential equation (SDE) or Langevin equation[14] for the stochastic variable $\gamma_i(t)$ which will depend on a particular realization of a set of Gaussian white noises $\{\xi_n(t)\}$. There are conceptual problems involved with the notion of Brownian motion in superspace, but we draw the reader's attention to recent work[15] which has placed this subject on a sound mathematical footing. Our approach, however, will not draw upon any concepts of Grassmann stochastic variables—rather, we shall invoke the philosophy that any SDE which reproduces the same *average* quantities as the corresponding Fokker–Planck equation is a valid description of the physical process. This philosophy is clear when one realizes that there is no unique SDE corresponding to a given Fokker–Planck equation; for instance, the noise term

that appears in a SDE is only ever defined up to an orthogonal transformation[14] [see (3.16) below]. To be more specific concerning this point, we shall insist that averages of the form (3.9)—when calculated from the SDE—are zero, whereas we demand no such constraint for any given realization of this quantity. One of the consequences of this approach is that the noise appearing in the SDE will be the usual *real-valued* white noise.

Following the usual correspondence between the Fokker–Planck equation and the SDE (in the Ito interpretation), we can write down the SDE related to (3.12) in the form

$$d\gamma_i(t) = A_i(\gamma)\, dt + \sum_j g_{i,j}(\gamma)\, dW_j(t) \tag{3.16}$$

where $g_{i,j}$ is related to the diffusion matrix via

$$B_{i,j}(\gamma) = \sum_k g_{i,k}(\gamma)\, g_{j,k}(\gamma) \tag{3.17}$$

and $\{W_j(t)\}$ are independent Wiener processes related to the usual Gaussian-distributed white noise functions via $dW_j(t) = \xi_j\, dt$. The existence of the matrix $g_{i,j}$ is not clear *a priori*, but it can be given explicitly for many examples. Often one may proceed to calculate quantities of interest using the diffusion matrix only, so that the matrix $g_{i,j}$ appears as a (natural) intermediate step. We can check if this SDE is consistent with (3.12) by calculating equations of motion for correlation functions from (3.16) with the use of the Ito formula (which tells us how to change variables in Ito calculus). Because of the Grassmann character of the variables, the derivatives appearing in the Ito formula pick up a minus sign [see (3.2)]. So, given a function $F(\gamma)$, we can determine the SDE appropriate to F by using the (slightly altered) Ito formula:

$$\begin{aligned} dF(\gamma) = &-\sum_n A_n(\gamma)\, \partial_{\gamma_n} F(\gamma)\, dt + \tfrac{1}{2} \sum_{n,m} B_{n,m}(\gamma)\, \partial_{\gamma_n} \partial_{\gamma_m} F(\gamma)\, dt \\ &- \sum_{n,m} g_{n,m}(\gamma)\, \partial_{\gamma_n} F(\gamma)\, dW_m(t) \end{aligned} \tag{3.18}$$

Use of (3.16) and (3.18) reproduces the equations of motion for correlation functions that were previously obtained from both (2.1) and (3.13).

There is one small complication which must be addressed before we can use (3.16) with confidence. If $F(\gamma) = \gamma_i^2$, then (3.18) reduces to the trivial identity $0 = 0$. We must be sure that calculating any moment of the form (3.9) by integrating (3.16) and averaging over the noise gives zero. It

turns out that this condition may be completely satisfied [for all moments of the form (3.9) and for all times] by making a special choice for the diagonal part of the diffusion matrix (which up until now has been arbitrary). For consistency we require

$$B_{i,i}(\gamma) = -2\gamma_i A_i(\gamma) \tag{3.19}$$

where A_i is the drift vector.

Therefore for the traffic model we have an exact SDE of the form (3.16) where $A_i(\gamma)$ is given by (3.14) and $g_{i,j}(\gamma)$ is obtained from (3.17) with the particular form of the diffusion matrix

$$B_{i,j}(\gamma) = \gamma_i \gamma_j (\delta_{i,j-1} + \delta_{i,j+1}) - 2\gamma_i A_i(\gamma)\, \delta_{i,j} \tag{3.20}$$

This completes our program of deriving an exact stochastic description for exclusion processes. In the next section we shall present a solution of the SDE for the traffic model (with $a=0$) as a simple illustration of the validity of the formalism.

4. SOLUTION OF THE SDE FOR $a=0$

From the last section we can write down an exact SDE for the traffic model, with the corresponding RSOS interface being initially flat. In the case of $a=0$ this equation has the form

$$\dot{\gamma}_i = -\gamma_i + \frac{1}{2}(\gamma_{i-1} + \gamma_{i+1}) + \frac{1}{2^{1/2}}[(\gamma_i - \gamma_{i-1})\,\xi_i + (\gamma_i - \gamma_{i+1})\,\xi_{i+1}] \tag{4.1}$$

As we mentioned earlier, the traffic model is intimately related to the single-step RSOS model [in $(1+1)$ dimensions]. In the Appendix we shall prove that the width of the interface $W(t)$ for the (unbiased) RSOS model is related to the two-point correlation function $c_1(t)$ of the traffic model (with $a=0$) via

$$\frac{d(W^2(t))}{dt} = 2 - 8c_1(t) \tag{4.2}$$

where $c_n(t) \equiv \langle x_i x_{i+n} \rangle$.

There are two ways to calculate c_1. First one can find c_1 by solving the hierarchy of correlation functions which may be derived from (2.1), (3.13) or from (4.1). Alternatively one can just integrate (4.1) and calculate c_1

explicitly. The second method is simple and one obtains the following equation for c_1:

$$c_1(t) = \tfrac{1}{2}e^{-2t}[\sinh(2t) - I_1(2t)] + \int_0^t dt'\, c_1(t-t')\, e^{-2t'}\, [I_0(2t') - 2I_1(2t') + I_2(2t')] \tag{4.3}$$

where I_n is the modified Bessel function.[16] This is easily solved by Laplace transform to give

$$c_1(t) = \tfrac{1}{4} - \tfrac{1}{4}\, e^{-2t} I_0(2t) \tag{4.4}$$

This result agrees with that obtained via the method of solving the hierarchy of correlation functions, which indicates the self-consistency of our formulation of exact SDEs for exclusion processes.

Combining (4.2) and (4.4) and using the asymptotic form of the modified Bessel function (for large argument),[16] we have

$$W(t) = \left(\frac{4}{\pi}\right)^{1/4} t^{1/4}[1 + O(t^{-1})] \tag{4.5}$$

for large times. This is the expected result for fluctuations in an unbiased interface.

5. SUMMARY AND CONCLUSIONS

We have presented a new formulation of exclusion processes. The method is completely general and leads to descriptions of the process in the form of exact Fokker–Planck and Langevin equations (and also field theories through use of, for instance, the MSR[17] method as applied to the SDE.) We have demonstrated the method for the one-dimensional traffic model for ease of presentation. In the final section we solved this model (in the unbiased case) using the exact SDE, and through the mapping from the traffic model to the single-step RSOS model in $(1+1)$ dimensions, we obtained the expected result of $W(t) \sim t^{1/4}$ for an unbiased interface (or, in fact, for an unbiased directed polymer).

We think that this method may be of use in the study of various interface growth models in $(2+1)$ dimensions, where the existence of universality is uncertain, forcing one to consider different models in more detail.

APPENDIX

In this short appendix we use the mapping from the single-step RSOS in $(1+1)$ dimensions to the traffic model in one dimension, in order to relate the fluctuations of the interface in the former model to the space correlations in the latter—hence deriving Eq. (4.2). In Fig. 1 we show the simple mapping of the single-step interface onto a spin model. Transitions in the interface correspond to spin exchange. The traffic model is identical to the spin model with up/down spins corresponding to occupied/vacant sites. An initially flat interface corresponds to alternately occupied sites in the traffic model. From this mapping it is clear that

$$h_i = h_0 + \sum_{m=0}^{i-1} (2x_m - 1) \tag{A.1}$$

where h_i is the height of the interface in column i. Defining the mean square width (between sites i and j) as

$$w(|i-j|, t)^2 = \langle (h_i - h_j)^2 \rangle_P \tag{A.2}$$

we find that

$$w(n, t)^2 = 8 \sum_{m=1}^{n-1} A(n-m, t) + n(2-n) \tag{A.3}$$

where

$$A(n, t) = \sum_{r=1}^{n} \langle x_k x_{k+r} \rangle_P = \sum_{r=1}^{n} c_r(t) \tag{A.4}$$

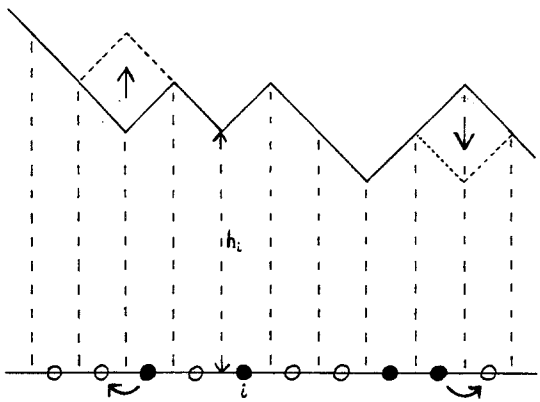

Fig. 1. The mapping from the single-step RSOS model to the traffic model. Transitions in the interface correspond to the (restricted) diffusion of particles in the traffic model.

The equations of motion for two-point correlators [which are easily derived from (2.1), (3.13), or (4.1)] have the form

$$\partial_t c_1(t) = -c_1(t) + c_2(t) \tag{A.5}$$

and for $n \geqslant 2$

$$\partial_t c_n(t) = -2c_n(t) + c_{n-1}(t) + c_{n+1}(t) \tag{A.6}$$

This implies that

$$\partial_t A(n, t) = c_{n+1}(t) - c_n(t) \tag{A.7}$$

So, differentiating (A.3) with respect to t and using (A.7) then gives

$$\frac{d(w(n, t)^2)}{dt} = 8[c_n(t) - c_1(t)] \tag{A.8}$$

Finally, taking $n \to \infty$, we have (4.2), where we have used $c_\infty = \frac{1}{4}[1 + O(e^{-2t})]$ and have defined $W(t) = \lim_{n \to \infty} w(n, t)$.

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REFERENCES

1. F. Family and D. P. Landau, eds., *Kinetics of Aggregation and Gelation*, (North-Holland, Amsterdam, 1984), and references therein.
2. M. Eden, in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, F. Neyman, ed. (University of California, Berkeley, 1961), Vol. IV.; J. G. Zabolitzky and D. Stauffer, *Phys. Rev. A* **34**:1523 (1986).
3. M. J. Vold, *J. Colloid Interface Sci.* **18**:684 (1963); D. N. Sutherland, *J. Colloid Interface Sci.* **22**:300 (1966); P. Meakin, *J. Colloid Interface Sci.* **105**:240 (1985).
4. J. M. Kim and J. M. Kosterlitz, *Phys. Rev. Lett.* **62**:2289 (1989).
5. T. A. Witten and L. M. Sander, *Phys. Rev. Lett.* **47**:1400 (1982).
6. M. Kardar, G. Parisi, and Y.-C. Zhang, *Phys. Rev. Lett.* **56**:889 (1986).
7. M. Kardar and Y.-C. Zhang, *Phys. Rev. Lett.* **58**:2087 (1987).
8. P. Meakin, P. Ramanlal, L. M. Sander, and R. C. Ball, *Phys. Rev. A* **34**:5091 (1986).
9. B. Grossmann, H. Guo, and M. Grant, *Phys. Rev. A* **43**:1727 (1991).
10. D. A. Huse, J. G. Amar, and F. Family, *Phys. Rev. A* **41**:7075 (1990).
11. G. Parisi and Y. C. Zhang, *Phys. Rev. Lett.* **53**:1791 (1984); D. Elderfield, *J. Phys. A* **18**:L773 (1985).

12. C. W. Gardiner and S. Chaturvedi, *J. Stat. Phys.* **17**:429 (1977).
13. J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford, 1989).
14. C. W. Gardiner, *Handbook of Stochastic Methods*, 2nd ed., (Springer-Verlag, Berlin, 1985).
15. A. Rogers, *J. Phys. A* **25**:447 (1992).
16. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, 10th ed. (Dover, New York, 1972).
17. P. C. Martin, E. D. Siggia, and H. A. Rose, *Phys. Rev. A* **8**:423 (1978); C. De Dominicis and L. Peliti, *Phys. Rev. B* **18**:353 (1978).